



TITLE:

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CITATION:

Okuyama, Yusuke. A generalization of the Cremer theorem by the Nevanlinna theoretical argument (Complex Dynamics). 数理解析研究所講究録 2005, 1447: 170-175

ISSUE DATE:

2005-08

URL:

<http://hdl.handle.net/2433/47659>

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A generalization of the Cremer theorem by the Nevanlinna theoretical argument

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2 July, 2003

This article is a resume of the paper [11]. We study an irrationally indifferent cycle of points or circles of a rational function, which is either Siegel or Cremer by definition. We give a clear interpretation of some Diophantine quantity associated with an irrationally indifferent cycle as a quantity arising in the Nevanlinna theory. As a consequence, we show that an irrationally indifferent cycle is Cremer if this Nevanlinna-theoretical quantity does not vanish, which generalize the classical Cremer condition.

Let f be a rational function of degree ≥ 2 and $f^k := f^{\circ k}$ for $k \in \mathbb{N}$. $F(f)$ and $J(f)$ denote the Fatou and Julia sets of f respectively. The classification of cyclic Fatou components D is known: The pair (g, D) , where g is the first return map of f on D , is an *attractive basin* if $\{g^n\}_{n=1}^\infty$ converges to a point in D locally uniformly on D . The *parabolic basin* is similar, but $\{g^n\}_{n=1}^\infty$ converges to a point in the boundary of D locally uniformly on D . When g is a proper selfmap of D of degree ≥ 2 , (g, D) is one of them. When g is a univalent selfmap of D , D is called a *rotation domain* since (g, D) is conformally conjugate to an irrational rotation on either a disk or an annulus, and called a *Siegel disk* or an *Herman ring* in each of cases. For the details, see [9], [1], [3], [8].

Our main interest is an irrationally indifferent cycle of points or circles.

Definition 1 (irrationally indifferent cycle of points or circles). A point z_0 in $\hat{\mathbb{C}}$ is *periodic* if for some $p \in \mathbb{N}$, $f^p(z_0) = z_0$. The least such p is the *period* of z_0 .

*Partially supported by the Sumitomo Foundation, and by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Young Scientists (B), 15740085, 2003

$\{f^n(z_0)\}_{n=1}^p$ is a cycle of points, and $\lambda := (f^p)'(z_0)$ is the multiplier of it. This cycle of points is *irrationally indifferent* if $\lambda = e^{2\pi i\alpha}$ for some $\alpha \in \mathbb{R} - \mathbb{Q}$.

A topological circle $S \subset \hat{\mathbb{C}}$ is *periodic* if for some $p \in \mathbb{N}$, $f^p(S) = S$ and $f^p|_S : S \rightarrow S$ is an orientation-preserving homeomorphism. The least such p is the *period* of S , $\{f^n(S)\}_{n=1}^p$ is a cycle of circles, and $\lambda := e^{2\pi i\alpha}$ is the multiplier of it, where $\alpha \in \mathbb{R}/\mathbb{Z}$ is the *rotation number* (cf. [5]) of a S^1 -homeomorphism ϕ which is topologically conjugate to $f^p|_S$. This cycle of circles is *irrationally indifferent* if α is irrational.

It is known that if irrationally indifferent cycles of points or circles intersect $F(f)$, then they are contained in some rotation domains.

Definition 2 (Siegel and Cremer cycles). An irrationally indifferent cycle of points or circles is a *Siegel cycle* if it is contained in $F(f)$. Otherwise it is a *Cremer cycle*.

The following is an unsolved problem with a long history: *Given an irrationally indifferent cycle of points or circles, how can we judge whether it is contained in the Fatou set or not?*

The following answers the problem for points in one direction.

Theorem 1 (Siegel, Brjuno, Rüssmann and Yoccoz). Every analytic germ $f(z) = \lambda z + O(z^2)$, $\lambda = e^{2\pi i\alpha}$ ($\alpha \in \mathbb{R} - \mathbb{Q}$), at the origin is analytically linearizable if λ satisfies the Brjuno condition, which is a Diophantine(-type) condition.

For the precise definition of Brjuno condition, see [2] and [17]. In the reverse direction,

Theorem 2. Let P be a quadratic polynomial. An irrationally indifferent cycles of points of P is Cremer if its multiplier does not satisfy the Brjuno condition.

Remark 1. Theorem 2 is proved by Yoccoz ([17]) in the case of period one, and, from this, later generalized by the author ([10]) in the case of arbitrary periods.

Until now, only for quadratic polynomials and only for points, the complete answer of the problem is known. The classical Cremer Theorem is a partial answer for the problem for points in the reverse direction:

Theorem 3 (Cremer[4] (1932)). Let f be a rational function of degree $d \geq 2$, and O be an irrationally indifferent cycle of points of period p and of multiplier λ . O is Cremer if λ satisfies

$$\limsup_{n \rightarrow \infty} \frac{1}{d^{pn}} \log \frac{1}{|\lambda^n - 1|} = \infty. \quad (1)$$

We succeeded in improving the Cremer condition (1) and extending his result to irrationally indifferent cycles of circles.

Main Theorem 1 (Criterion for Cremer [11]). *Let f be a rational function of degree $d \geq 2$, and O be an irrationally indifferent cycle of either points or circles of period p and of multiplier λ .*

O is Cremer if λ satisfies

$$\limsup_{n \rightarrow \infty} \frac{1}{d^{pn}} \log \frac{1}{|\lambda^n - 1|} > 0. \quad (2)$$

This improvement is substantial. In fact, For every $x \in [0, \infty]$, it is easy to find $\alpha \in \mathbb{R} - \mathbb{Q}$ such that the left hand side of (2) equals x (cf. [6]).

Remark 2. In the case of polynomials and irrationally indifferent cycle of points, Pierre Tortrat ([16]) showed a similar result to Main Theorem 1 by using a potential theoretical argument. Indeed, his condition, which is slightly technical, coincides with ours.

Main Theorem 1 naturally follows from the interpretation of the left hand side of (1) as a *Nevanlinna-theoretical* quantity, and its *vanishing theorem*.

Let $[p, q]$ be the chordal distance between $p, q \in \hat{\mathbb{C}}$ such that $[0, \infty] = 1$, and σ the spherical area measure on $\hat{\mathbb{C}}$ such that $\sigma(\hat{\mathbb{C}}) = 1$. We write the set of all rational endomorphism of $\hat{\mathbb{C}}$ by Rat , and call a sequence in Rat a *rational sequence*.

The following tools of the Nevanlinna theory for rational sequences was invented by Sodin in [15].

For $f, g \in \text{Rat}$, we define the *pointwise proximity function*:

$$(w(g, f))(z) := \log \frac{1}{[g(z), f(z)]} : \hat{\mathbb{C}} \rightarrow [0, \infty],$$

and the *mean proximity*:

$$m(g, f) := \int_{\hat{\mathbb{C}}} w(g, f) d\sigma \in [0, \infty).$$

For a rational sequence $\mathcal{F} = \{f_k\}_{k=1}^{\infty}$, we define the *characteristic sequence* $\{T_{\mathcal{F}}(k) := \deg f_k\}_{k=1}^{\infty} \subset \mathbb{N}$.

Using the above tools, we shall define some deficiency of a rational sequence with the increasing characteristic sequence:

Definition 3 (cf. [7]). Let $\mathcal{F} = \{f_k\}_{k=1}^{\infty}$ be a rational sequence with the increasing characteristic sequence. For $g \in \text{Rat}$, we define the *Valiron exceptionality*:

$$\text{VE}(g; \mathcal{F}) := \limsup_{k \rightarrow \infty} \frac{m(g, f_k)}{T_{\mathcal{F}}(k)} \in [0, \infty].$$

Although the *Valiron exceptionality* is defined by a quite complicate way, we can treat it by the following.

Main Theorem 2 (Fundamental Equality). *Let $f \in \text{Rat}$ be of degree $d \geq 2$, and $g \in \text{Rat}$ not identically equal to such $z \in \hat{\mathbb{C}}$ as $\# \bigcup_{n \in \mathbb{N}} f^{-n}(z) < \infty$.*

Then for every positive continuous function $\phi \not\equiv 0$ on $\hat{\mathbb{C}}$,

$$\text{VE}(g; \{f^k\}) = \limsup_{k \rightarrow \infty} \frac{\int_{\hat{\mathbb{C}}} \phi \cdot w(g, f^k) d\sigma}{d^k \cdot \int_{\hat{\mathbb{C}}} \phi d\sigma}.$$

By the fundamental equality, we obtain two Main Theorems.

Main Theorem 3 (Vanishing Theorem). *Let f be a rational function of degree ≥ 2 such that $F(f) \neq \emptyset$. Then $\text{VE}(\text{Id}_{\hat{\mathbb{C}}}, \{f^k\}) = 0$.*

Problem. Does $\text{VE}(\text{Id}_{\hat{\mathbb{C}}}, \{f^k\})$ always vanish even if $J(f) = \hat{\mathbb{C}}$?

Main Theorem 4 (Natural Equality). *Let O be an irrationally indifferent cycle of either points or circles of period p and of multiplier λ . If O is Siegel, then*

$$\limsup_{k \rightarrow \infty} \frac{1}{d^{pk}} \log \frac{1}{|\lambda^k - 1|} = \text{VE}(\text{Id}_{\hat{\mathbb{C}}}, \{f^{pk}\}). \quad (3)$$

Now the proof of Main Theorem 1 is straightforward:

Proof of Main Theorem 1 If O is Siegel, then $F(f) \neq \emptyset$. Hence from the Vanishing Theorem, $\text{VE}(\text{Id}_{\hat{\mathbb{C}}}, \{f^{pk}\}) = 0$, so by the Natural equality, the left hand side of (3) vanishes. \square

Remark 3. Our method of studying multipliers of irrationally indifferent cycles unifies the cases both of points and of circles, and in particular, dispenses with such quasiconformal surgeries as in [12] for Siegel cycles of circles. However, when this cycle of circles is contained in an Herman ring, by quasiconformal surgery of it (cf. [12], [13] and [14]), we obtain a rational function \tilde{f} whose degree is less than that of f and which has a Siegel disk with the same rotation number as the original Herman ring of f . Hence by applying Main Theorem 1 to \tilde{f} rather than f , a stronger conclusion than Main Theorem 1 follows.

ACKNOWLEDGEMENT. This work was partially done while the author was a long term researcher of International Project Research 2003 “Complex Dynamics” of RIMS of Kyoto University. The author is very grateful to Prof. Mitsuhiro Shishikura, who is the chair of the project, and the staff of RIMS of Kyoto University for their hospitality.

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